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Frequency-Domain Modeling and Analysis of Stochastic Flows in Transportation Networks

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Abstract

In this paper, we study the transportation network flow problem with stochastic flow distribution. According to the stochastic properties of transportation network flows, we first discuss the functional relations between the arcs and the nodes for modeling the transportation network systems, and originally construct an incidence matrix of capacity in the frequency domain. Then a new frequency-domain spanning graph model to analyze the network stochastic maximum flow and the stochastic minimum flow problem is presented. A corresponding algorithm is designed to resolve the model. In our solution, through the mutual transformations of probability functions between time-domain and frequency-domain in our model, the dynamic process of origin-destination flows is analyzed qualitatively; hence the minimum flow and maximum flow are derived quantitatively. An advantage of our approach is that it can handle uniformly both the continuous probability distribution case and the discrete probability distribution case. Numerical experiment results illustrate the feasibility and effectiveness of our approach.

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Keywords: Combinatorial optimization; stochastic networks; stochastic max flows; stochastic min flows; frequency-domain analysis

1. Introduction and related work

The maximum/minimum flow problem is defined on a directed network graph $G=(V,E), O\in V, D\in V$ with nodes $1,2,\cdots,n$ and a nonnegative capacity c_{ij} assigned to each arc, where O is the origin and D is the destination. It is to find the maximum/minimum possible flow from the specified origin O to the specified destination D without violating any constraints on the arc capacities. One of the fundamentals (Maximum-flow Min-cut Theorem), which results in combinatorial optimization is that the value of maximum flow equals the capacity of minimum cut.

Since Ford and Fulkerson presented the celebrated maximum flow algorithm of 2F in 1957 [1] and developed a labeling procedure to find a flow pattern whose flow $F_{O,D}$ from O (origin, or source) to D (destination) is maximal in 1962 [2], many researchers have done extensive work on network flow theory. Many efficient algorithms have been developed based on Maximum-Flow Min-Cut theorem by researchers including Ford [2], Edmonds and Karp [3], Dinitz [4], Karzanov [5], Malhotra [6], Galil [7][8], Ahuja [9]. However, in these researches, the capacities of

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arcs or nodes are always assumed to be constant in the weighted network models. For many practical and modern transportation networks, due to factors such as traffic accidents, road crowding, weather variability, various kinds of uncertainties are frequently encountered. These factors result in uncertainties in the arc and node capacities. A typical solution to deal with uncertainties in these situations is finding the expected maximum/minimal flow. However, this approach does not make much sense in many practical situations. For example, for the reliability problem of network maximum flow capacity and minimum flow capacity, decision-makers are more interested in the probability distribution of the reliability than the expected values. At present, there are not many solutions on the network flow with stochastic arc capacity: the primary stochastic models and algorithms are developed through expected value method. The research on obtaining the probability distribution is very few. Kim and Roush [10] presented a theory of fuzzy flows and studied closed formulas giving necessary and sufficient conditions for admissible flows to exist and how to get the maximal admissible flow. Jane et al [11] studied the distribution of maximal flow to evaluate the probability. Lin [12][13] and Yeh [14] presented algorithms to generate all d -minimal paths (or named lower boundary points for *d*) for the demand d in order to evaluate a performance index, with the probability that the system capacity is larger than or equal to d . Afterwards, Lin et al [15][16] and Koetter et al [17] developed methods to evaluate the performance index for a network that allows multiple commodities to be transmitted from origin to destination. Ning [18] studied the minimum flow problem in transportation network and its approximate solution method. For solution methods based on expected value, they have the advantage of simplicity, however, pertinent information about the stochastic transportation network (such as higher-order moments of the arc distributions) are lost. In contrast, methods based on probability distribution can reflect the comprehensive information of transportation state. Existing solution methods based on probability distribution usually deal with the models with continuous probability distribution and discrete probability distribution separately. Therefore, it is important to study new analysis methods in order to analyze network flow problem in a uniform framework. In this paper, we propose a new frequency-domain approach for stochastic transportation network. It is capable to analyze network stochastic maximum flow and stochastic minimum flow, at the same time handle both cases of continuous probability distribution and discrete probability distribution.

The remainder of this paper is organized as follows. Section 2 presents our frequency-domain spanning graph model, and describes the definitions and one important lemma for analyzing the stochastic minimum flow and stochastic maximum flow problem in any transportation network. Section 3 describes the design of our frequency-domain solution algorithm. Section 4 gives a numerical example to demonstrate the feasibility and effectiveness of our approach. Finally Section 5 summarizes the paper with conclusions.

2. Problem formulation and modeling

In this section, based on the traditional model when the capacity of each arc in the transportation network is deterministic, we present the frequency-domain spanning graph model with the stochastic probability to model the capacity uncertainties. In order to analyze the problems of stochastic minimum and maximum flow, four definitions and one lemma are detailed. Especially an incidence matrix of capacity in the frequency domain is originally constructed to represent and analyze the functional relation between the arcs and the nodes. According to these descriptions, we propose a mathematic formulation of the frequency-domain model to analyze the capacity of the stochastic flow.

Frequency-domain Spanning Graph (FSG) is from the classical graph theory model (node-line model) in the time-domain, through the mutual transformations of probability functions between time-domain and frequency-domain, finally is a spanning block diagram model in the frequency-domain. In order to formulate the frequency-domain spanning graph model for stochastic transportation network flow based on dynamic road conditions, we give a directed spanning network $G=(V,E)$ with an origin node v_O and a destination node v_D as shown in Fig. 1. $V=\{v_i(s)\}$ is defined as a finite set of nodes and their frequency-domain capacity states (which represents traffic hinges, cities, towns and special nodes etc.), $E=\{e_{ij}(s)\}$ is defined as a set of arcs and their frequency-domain capacity states (which represents sections of highway or roadway etc.).

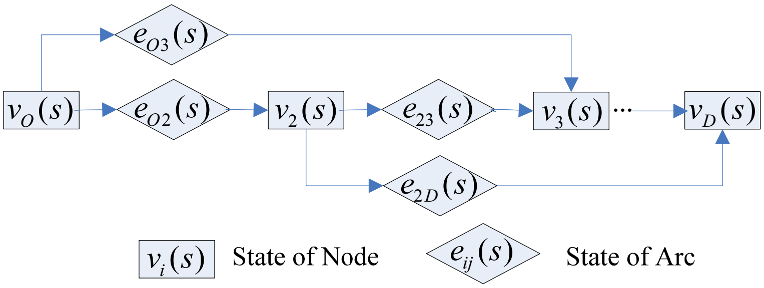

Fig. 1. Frequency-domain capacity model of stochastic transportation network.

Due to different kinds of uncertainties discussed in Section 1, arc capacity c_{ij} is represented as a random variable in the time domain, whose probability density is $f_{ij}(c)$ (may be continuous or discrete); node capacity c_i is only affected by the related arcs (ignored the self-capacity) [19]. c_i is also a random variable.

Definition 1:

$$e_{ij}(s) = E[e^{-sC}] = \begin{cases} \int_0^\infty e^{-sc} f_{ij}(c)dc & \text{Continuous distribution} \\ \sum_{\text{all c}} e^{-sc} p_{ij}(c) & \text{Discrete distribution} \end{cases}, \tag{1}$$

$$v_i(s) = g(e_{ij}(s), e_{ji}(s)). \tag{2}$$

where in Equation (1), $e_{ij}(s)$ denotes the probability density of arc e_{ij}'s capacity in the frequency domain. For example $e_{ij}(s) = e^{\frac{1}{2}\sigma_{ij}^2 s^2 - \mu_{ij} s}$ can be used to describe the probability density of arc e_{ij}'s capacity that conforms the Normal distribution; and its inverse Laplace transform is $f_{ij}(c) \sim N(\mu_{ij}, \sigma_{ij}^2)$.

Sometimes since the probability density of the arc capacity may be influenced by the uncertainties, it is difficult to obtain an accurate continuous probability density. In such a case, we can only get a discrete probability density by the qualitative analysis and empirical evaluations. For example when the occurrence probabilities of 3, 4, and 5 units' capacity are equal to 0.3, 0.5, and 0.2 respectively, its discrete probability density can be depicted by $e_{ij}(s) = 0.3e^{-3s} + 0.5e^{-4s} + 0.2e^{-5s}$. As in Equation (2), $v_i(s)$ denotes the probability density of node v_i's capacity in the frequency domain. It is a function of arcs $e_{ij}(s)$ and $e_{ji}(s)$, in which $e_{ij}(s)$ is the frequency-domain density function of node v_i's outflow arcs, whereas $e_{ji}(s)$ is the inflow arcs' function.

Definition 2: The capacity Δ_i of node v_i is equal to the value of the total capacity of node v_i's outflow directed arcs minus that of node v_i's inflow directed arcs. It can be expressed as the following equation:

$$\Delta_i = \sum_{v_j \in V} c_{ij} - \sum_{v_j \in V} c_{ji} = c_i^+ - c_i^-. \tag{3}$$

We also define

$\int_{-\infty}^0 f_i(\Delta) d\Delta_i > \int_0^\infty f_i(\Delta) d\Delta_i$: The stochastic negative degree, i.e. $E(\Delta_i) < 0$;

$\int_{-\infty}^0 f_i(\Delta) d\Delta_i \le \int_0^\infty f_i(\Delta) d\Delta_i$: The stochastic positive degree, i.e. $E(\Delta_i) \ge 0$.

Definition 3: In a directed transportation network G, the incidence matrix of capacity in the frequency domain is defined as follows,

$$
\begin{array}{cccc}
\text{Nodes} & \text{States of outflow} & \text{States of inflow} & \text{Stochastic degree} \\
\begin{array}{c} v_O \\ \vdots \\ v_i \\ \vdots \\ v_D \end{array} &
\multicolumn{2}{c}{\begin{bmatrix} \prod\limits_{v_j \in V} e_{Oj}(s) & 1 \\ \vdots & \vdots \\ \prod\limits_{v_j \in V} e_{ij}(s) & \prod\limits_{v_j \in V} e_{ji}(-s) \\ \vdots & \vdots \\ 1 & \prod\limits_{v_j \in V} e_{jD}(-s) \end{bmatrix}} &
\begin{array}{c} E(\Delta_O) \\ \vdots \\ E(\Delta_i) \\ \vdots \\ E(\Delta_D) \end{array}
\end{array} . \tag{4}
$$

As it shows, the first column of the incidence matrix stands for nodes in the network; the second column stands for the capacity states of the corresponding node's outflow arcs, and is also the product of the node v_i's outflow arcs $e_{ij}(s)$; the third column stands for the capacity states of the corresponding node's inflow arcs, and is also the product of the node v_i's inflow arcs $e_{ji}(s)$; the fourth column stands for the stochastic degree proposed in Definition 2. For the nodes without any outflow arcs or inflow arcs, the corresponding state equals to 1 in the frequency domain.

Definition 4: In a directed transportation network G, we divide node set V into two non-empty sets S and $\overline{S}$ (where $S \bigcup \overline{S} = V, S \bigcap \overline{S} = \phi,\ v_O \in S, v_D \in \overline{S}$). The corresponding arc set $(S,\overline{S}) = \{[e_{ij} \mid v_i \in S, v_j \in \overline{S}], [e_{ij} \mid v_i \in \overline{S}, v_j \in S],\ e_{ij} \in E\}$ is defined as the bi-directional cut-set of the transportation network. The set of directed arcs pointing from S to $\overline{S}$ in the bi-directional cut-sets is defined as the positive cut-set, i.e. $(S,\overline{S})^+$; otherwise, the negative cut-set, i.e. $(S,\overline{S})^-$; the capacity of the bi-directional cut-set $(S,\overline{S})$ is the capacity of the positive cut-set $(S,\overline{S})^+$ minus the capacity of the negative cut-set $(S,\overline{S})^-$, i.e. $C(S,\overline{S}) = C(S,\overline{S})^+ - C(S,\overline{S})^- = \sum\limits_{v_i \in S, v_j \in \overline{S}} c_{ij} - \sum\limits_{v_i \in \overline{S}, v_j \in S} c_{ij}$.

When the directed flows from $\overline{S}$ to S are zero, the transportation cut-section reaches the maximum flow capability, that is $C(S,\overline{S}) = C(S,\overline{S})^+$; when the directed flows from $\overline{S}$ to S gradually increase, the flow capability of the transportation cut-section reduces correspondingly; when the capacity of the directed flows equals to $C(S,\overline{S})^-$, the transportation cut-section reaches the minimum flow capability, that is $C(S,\overline{S}) = C(S,\overline{S})^+ - C(S,\overline{S})^-$.

Therefore the minimum value of flow capacity in each positive cut-set $(S,\overline{S})^+$ is the max-flow; the minimum value of flow capacity in each bi-directional cut-set $(S,\overline{S})$ is the min-flow. Similarly, the second minimum value of flow capacity is the 1th min-flow; the third minimum value of flow capacity is the 2th min-flow, etc.

Lemma: In a directed transportation network G, the capacity of an arbitrary bi-directional cut-set $(S,\overline{S})$ equals to the sum of all nodes' capacities in the node set S. We give the following equation:

$$C(S,\overline{S}) = \sum_{v_i \in S} \Delta_i . \tag{5}$$

We define the minimum bi-directional-cut-set $(S,\overline{S})_{\min}$ (i.e. min-flow), where node set S is consisted of nodes v_i with $E(\Delta_i) < 0$ (the origin node $v_O \in S$ though $E(\Delta_O) > 0$), and node set $\overline{S}$ is consisted of nodes v_i with $E(\Delta_i) \geq 0$ (the destination node $v_D \in \overline{S}$ though $E(\Delta_D) < 0$). The minimum positive cut-set $(S,\overline{S})^+_{\min}$ (i.e. max-flow) is the minimum sum of $C(S,\overline{S})$ and $C(S,\overline{S})^-$.

Proof: Suppose that $v_i (i = 1, 2, \cdots, r)$ are all nodes in set S, and $v_j (j = n-r, \cdots, n)$ are all nodes in set $\overline{S}$. According to Definition 4 it is easy to deduce:

$$C(S,\overline{S}) = \sum_{i=1}^{r} \sum_{j=n-r}^{n} c_{ij} - \sum_{j=n-r}^{n} \sum_{i=1}^{r} c_{ji},$$

$$\because \sum_{i=1}^{r}\sum_{j=1}^{r} c_{ij} = \sum_{j=1}^{r}\sum_{i=1}^{r} c_{ji},$$

$$\therefore C(S,\overline{S}) = \sum_{i=1}^{r}\sum_{j=n-r}^{n} c_{ij} - \sum_{j=n-r}^{n}\sum_{i=1}^{r} c_{ji} + \sum_{i=1}^{r}\sum_{j=1}^{r} c_{ij} - \sum_{j=1}^{r}\sum_{i=1}^{r} c_{ji}$$

$$= (\sum_{i=1}^{r}\sum_{j=n-r}^{n} c_{ij} + \sum_{i=1}^{r}\sum_{j=1}^{r} c_{ij}) - (\sum_{j=n-r}^{n}\sum_{i=1}^{r} c_{ji} + \sum_{j=1}^{r}\sum_{i=1}^{r} c_{ji})$$

$$= \sum_{i=1}^{r}(\sum_{j=n-r}^{n} c_{ij} + \sum_{j=1}^{r} c_{ij}) - \sum_{i=1}^{r}(\sum_{j=n-r}^{n} c_{ji} + \sum_{j=1}^{r} c_{ji})$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{n} c_{ij} - \sum_{i=1}^{r}\sum_{j=1}^{n} c_{ji} = \sum_{i=1}^{r} c_i^{+} - \sum_{i=1}^{r} c_i^{-} = \sum_{i=1}^{r} \Delta_i .$$

the capacity Δ_i of an arbitrary node (except the origin node v_O) in node set S might possibly be a stochastic positive degree. Hence, we can deduce as follows:

$$C(S,\overline{S}) = \sum_{v_i \in \forall S} \Delta_i = \sum_{i=1}^{l} \Delta_i + \sum_{i=1}^{r-l} \Delta_i \geq \sum_{i=1}^{k} \Delta_i = C(S,\overline{S})_{\min} . \tag{6}$$

where $v_i, i = 1,\cdots,l$ denotes some nodes with stochastic negative degree in node set V (including v_O), $v_i, i = 1,\cdots, r-l$ denotes some nodes with stochastic positive degree in V, and $v_i, i = 1,\cdots,k$ denotes all nodes with stochastic negative degree in V (including v_O). According to Equation (6), we propose that $C(S,\overline{S})_{\min}$ is equal to the sum of all nodes' capacities with $E(\Delta_i) < 0$ and the origin node's capacity with $E(\Delta_O) > 0$ in V (including v_O); in other words, all nodes v_i with stochastic negative degree belong to S.

According to Definition 4, $C(S,\overline{S}) = C(S,\overline{S})^{+} - C(S,\overline{S})^{-}$, we could easily deduce $C(S,\overline{S})^{+} = C(S,\overline{S}) + C(S,\overline{S})^{-}$. Q.E.D.

Inference: when S of $C(S,\overline{S})_{\min}$ decrease a node with stochastic negative degree $E(\Delta_i) < 0$, correspondingly the flow capability of the cut-section reduces. There will be the 1th min-flow or the 2th min-flow, etc.

In the frequency-domain spanning graph model, the capacities of the n arcs related to node v_i are random variables. Suppose $c_{i1}, c_{i2}, \cdots, c_{ir}$ are random variables which outflow from v_i, $c_{(r+1)i}, \cdots, c_{ni}$ are random variables that inflow to v_i; their corresponding frequency-domain functions $e(s)$ exist. Then the frequency-domain function of Δ_i is the product of n road arcs' capacity $e(s)$, or it could be regarded as the frequency-domain function of a node's capacity, we have:

$$v_i(s) = \Delta_i(s) = \prod_{j=1}^{r} e_{ij}(s) \prod_{j=r+1}^{n} e_{ji}(-s). \tag{7}$$

Deduction: Suppose that

$$\Delta_i = (c_{i1} + c_{i2} + \cdots + c_{ir}) - (c_{(r+1)i} + \cdots + c_{ni}),$$

Then according to Definition 1, the frequency-domain function of random variable Δ_i is

$$\Delta_i(s) = E[e^{-s[(c_{i1}+c_{i2}+\cdots+c_{ir})-(c_{(r\ 1)i}+\cdots+c_{ni})]}]$$

$$= E[\prod_{j=1}^{r} e^{-sc_{ij}} \prod_{j=r+1}^{n} e^{sc_{ji}}] = \prod_{j=1}^{r} E[e^{-sc_{ij}}] \prod_{j=r+1}^{n} E[e^{sc_{ji}}]$$

$$= \prod_{j=1}^{r} e_{ij}(s) \prod_{j=r+1}^{n} e_{ji}(-s) = v_i(s) = g(e_{ij}(s), e_{ji}(s)) .$$

In summary, the network flow problem of stochastic transportation is equivalent to finding the sum of all nodes' capacities in node set S which belongs to $(S,\overline{S})_{\min}$ in the frequency domain, i.e. the product of the above-mentioned nodes' frequency-domain functions. We give the mathematical model as:

$$q(s)=\prod_{v_i\in S} v_i(s), \tag{8}$$

$$v_i(s)=\prod_{v_j\in V} e_{ij}(s)\prod_{v_j\in V} e_{ji}(-s), \tag{9}$$

$$Q(s)=\frac{1-e^{-s}q(s)}{s}, \tag{10}$$

where Equation (8) is the frequency-domain probability density of the flow capacity in the stochastic transportation network. When S belongs to $(S,\overline{S})_{\min}$, $q(s)$ is the frequency-domain function of the min-flow; while S belongs to $(S,\overline{S})^{+}_{\min}$, $q(s)$ is the frequency-domain function of the max-flow. Equation (9) is the frequency-domain function of the node's capacity, and is also the product of node v_i 's outflow arcs $e_{ij}(s)$ and the inflow arcs $e_{ji}(-s)$. When $e_{ij}(s)$ multiplies $e_{ij}(-s)$, it is equivalent in expression as the random variables $c_{ij}-c_{ij}$ in the time domain, so $e_{ij}(s)e_{ij}(-s)=1$. Equation (10) is the frequency-domain probability distribution of the flow capacity, which facilitates in the comparison and analysis of the time-domain probability distribution curves after operating inverse Laplace transform.

Since in general it is difficult to analyze the probability density or the empirical distribution without mathematical function format [19][20], using conventional approaches in the time domain, we're considering using the frequency-domain spanning graph model to compute the flow capacity more efficiently. In such cases, we propose the following two important properties on the frequency-domain function:

Property 1: For an arbitrary probability distribution, if any order origin moment of its random variable exists, the frequency-domain function can be described as the series format as the following equation:

$$f_C(s)=\sum_{n=0}^{\infty}(-1)^n(\frac{s^n}{n!})E(C^n). \tag{11}$$

Proof: $\because\ f_C(s)=E(e^{-sC}),$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\cdots+(-1)^n\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}(-1)^n(\frac{x^n}{n!}),$$

$$\therefore\ f_C(s)=E[\sum_{n=0}^{\infty}(-1)^n\frac{(sC)^n}{n!}]=\sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}E[C^n].$$

Therefore, for any different probability distribution (especially the complex distributions), its frequency-domain function can be denoted as the series format utilizing this property, so as to carry out qualitative analysis and quantitative calculation, and computer simulations. For example, the negative exponential distribution is $f(c)=\lambda e^{-\lambda c}, c\ge 0$ in the time domain. According to Definition 1, we have:

$$f_C(s)=\int_0^{\infty}e^{-sc}\lambda e^{-\lambda c}dc=\int_0^{\infty}\lambda e^{-(\lambda+s)c}dc=-\frac{\lambda}{\lambda+s}e^{-(\lambda+s)c}\Big|_0^{\infty}=\frac{\lambda}{\lambda+s}.$$

So the frequency-domain function of the negative exponential distribution is $\frac{\lambda}{\lambda+s}$. Then we can obtain:

$$E(C)=-1\cdot\frac{\partial f_C(s)}{\partial s}\Big|_{s=0}=-\frac{\partial}{\partial s}(\frac{\lambda}{\lambda+s})\Big|_{s=0}=-\frac{-\lambda}{(\lambda+s)^2}\Big|_{s=0}=\frac{1}{\lambda},$$

$$E(C^2)=(-1)^2\cdot\frac{\partial^2 f_C(s)}{\partial s^2}\Big|_{s=0}=\frac{\partial^2}{\partial s^2}(\frac{\lambda}{\lambda+s})\Big|_{s=0}=\frac{2\lambda}{(\lambda+s)^3}\Big|_{s=0}=\frac{2}{\lambda^2},$$

$$\because \frac{\lambda}{\lambda+s}=\frac{1}{1+\frac{s}{\lambda}}, \quad \text{and} \because \quad \frac{1}{1+x}=1-x+x^2+\cdots+(-1)^n x^n+\cdots,$$

$$\therefore f_C(s)=\frac{\lambda}{\lambda+s}=1-\frac{s}{\lambda}+\frac{s^2}{\lambda^2}-\cdots+(-1)^n\frac{s^n}{\lambda^n}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}\cdot\frac{n!}{\lambda^n}=\sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}E(C^n).$$

In summary, $E(C^n)=\frac{n!}{\lambda^n}$, accordingly, $E(C)=\frac{1}{\lambda},E(C^2)=\frac{2}{\lambda^2},\cdots,E(C^n)=\frac{n}{\lambda^n}\cdots$, the frequency-domain function of the negative exponential distribution can be denoted as the series format to be simulated on the computer.

Property 2: The nth-order derivative for s of any frequency-domain function, where $s=0$, is equal to the product of nth-order origin moment of random variable C and the nth root of -1 in the time domain, i.e.:

$$f_C^n(s)\big|_{s=0}=(-1)^n E(C^n). \tag{12}$$

Proof: According to Property 1, we can calculate each order derivative for $f_C(s)$. Q.E.D.

The other proof is as follows,

$$\left.\frac{\partial f_C(s)}{\partial s}\right|_{s=0}=[\frac{\partial}{\partial s}\int_{-\infty}^{+\infty}e^{-sc}f(c)dc]_{s=0}=[\int_{-\infty}^{+\infty}(-c)\cdot e^{-sc}f(c)dc]_{s=0}=\int_{-\infty}^{+\infty}(-c)\cdot f(c)dc=-\int_{-\infty}^{+\infty}cf(c)dc=-E[C].$$

Similar with the above proof, we can get each order derivative exists where $s=0$. And this conclusion can be applied to the discrete random variables. Utilizing this property, it is simple to find the expected value and the variance of arc capacity's probability density.

Property 3: If there exist random variables $C_1,C_2,\cdots,C_n$, and also the corresponding frequency-domain functions $f_{C_i}(s)$ exist, then the frequency-domain function of the sum of random variables is identical to the product of the corresponding frequency-domain function of random variables.

Proof: Assume $X=C_1+C_2+\cdots+C_n$, according to Definition 1, the frequency-domain function of the random variable X is

$$f_X(s)=E[e^{-s(C_1+C_2+\cdots+C_n)}]=E[\prod_{i=1}^{n}e^{-sC_i}],$$

$\because C_1,C_2,\cdots,C_n$ are independent from each other, so we obtain

$$f_X(s)=E[\prod_{i=1}^{n}e^{-sC_i}]=\prod_{i=1}^{n}E[e^{-sC_i}].$$

3. Frequency-domain analysis algorithm

In this section, we propose a frequency-domain method to solve the stochastic max-flow and stochastic min-flow problems with the above-mentioned model.

STEP1: Obtain the incidence matrix of capacity in the frequency domain based on the frequency-domain spanning graph model, and then calculate the capacity of node v_i in the stochastic transportation network from Equation (9);

For example, based on Fig. 1, the incidence matrix is shown as follows:

$$\begin{array}{cccc}
\text{Nodes} & \text{States of outflow} & \text{States of inflow} & \text{Stochastic degree} \\
\begin{array}{c} v_O \\ v_2 \\ v_3 \\ \vdots \\ v_i \\ \vdots \\ v_D \end{array} &
\multicolumn{2}{c}{\begin{bmatrix} e_{O2}(s)\cdot e_{O3}(s) & 1 \\ e_{23}(s)\cdot e_{2D}(s) & e_{O1}(-s) \\ e_{34}(s) & e_{O3}(-s)\cdot e_{23}(-s) \\ \vdots & \vdots \\ e_{i\ i+1}(s) & e_{i-1\ i}(-s) \\ \vdots & \vdots \\ 1 & e_{2D}(-s)\cdot e_{n-1\ D}(-s) \end{bmatrix}} &
\begin{array}{c} E(\Delta_O) \\ E(\Delta_2) \\ E(\Delta_3) \\ \vdots \\ E(\Delta_i) \\ \vdots \\ E(\Delta_D) \end{array}
\end{array}.$$

$E(\Delta_O) = -e'_{O2}(s)\big|_{s=0} - e'_{O3}(s)\big|_{s=0} - 1' = -e'_{O2}(s)\big|_{s=0} - e'_{O3}(s)\big|_{s=0}$, $E(\Delta_2) = -e'_{23}(s)\big|_{s=0} - e'_{2D}(s)\big|_{s=0} - e'_{O1}(-s)\big|_{s=0}$,
$E(\Delta_3) = -e'_{34}(s)\big|_{s=0} - e'_{O3}(-s)\big|_{s=0} - e'_{23}(-s)\big|_{s=0}$, $E(\Delta_i) = -e'_{i\ i+1}(s)\big|_{s=0} - e'_{i-1\ i}(-s)\big|_{s=0}$, $E(\Delta_D) = -e'_{2D}(-s)\big|_{s=0} - e'_{n-1\ D}(-s)\big|_{s=0}$.

And according to Equation (9),
$v_O(s) = e_{O2}(s)\cdot e_{O3}(s)$, $v_2(s) = e_{23}(s)\cdot e_{2D}(s)\cdot e_{O1}(-s)$, $v_3(s) = e_{34}(s)\cdot e_{O3}(-s)\cdot e_{23}(-s)$, $v_i(s) = e_{i\ i+1}(s)\cdot e_{i-1\ i}(-s)$, $v_D(s) = e_{2D}(-s)\cdot e_{n-1\ n}(-s)$.

STEP2: Compare the stochastic positive or negative degree of every node from Step1. Then acquire $(S,\overline{S})_{\min}$, $(S,\overline{S})^{+}_{\min}$, k th- $(S,\overline{S})_{\min}$ and any desired cut sets according to the incidence matrix and the lemma proposed in Section 2; such as

1) The minimum bi-directional-cut-set $(S,\overline{S})_{\min}$ equals to the product of the corresponding nodes' frequency-domain functions in the incidence matrix, where the nodes belong to the node set S.

2) The minimum positive cut-set $(S,\overline{S})^{+}_{\min}$ equals to the product of the corresponding nodes' frequency-domain functions in the incidence matrix, where the nodes belong to the node set S and the $e_{ij}(-s)$ in the product equation must be removed.

STEP3: Obtain the frequency-domain function of the network's minimum flow capacity from Equation (8), which is identical to the product of all nodes' frequency-domain functions in node set S that belongs to $(S,\overline{S})_{\min}$; and then obtain the frequency-domain function of the network's maximum flow capacity, which is equivalent to the product of all nodes' frequency-domain functions for S that belongs to $(S,\overline{S})^{+}_{\min}$. $(S,\overline{S})_{\min}$ and $(S,\overline{S})^{+}_{\min}$ are the outcomes of STEP2;

STEP4: Use Equation (10) and the inverse Laplace transform to calculate the probability distribution of the flow capacity. Then plot the distribution curve graph based on the MATLAB [21]. The graph gives us an evident analysis of the differences of the min-flow, max-flow and other order of flows.

To the above algorithmic complexity of the computation, because convolution calculation in the time domain transform into multiplication in the frequency-domain, time and space complexity are reduced greatly. Its complexity is $O(mn)$, where m is the number of arc and n is the number of node in a network. The n nodes have the n operation, and the multiplication calculation of every node and their comparison do no exceed m time.

4. Numerical examples analysis

In order to analyze the stochastic maximum and minimum flows in a stochastic transportation network, we build its frequency-domain spanning graph model with an origin node $v_O = v_1$ and a destination node $v_D = v_6$, as shown in Fig. 2. The objective is to analyze the probability distribution of the max-flow when the transportation network is unblocked and the confidence level α is predetermined; as well as the probability distribution of the min-flow when the transportation network is blocked and the confidence level α is predetermined. We give the following analysis:

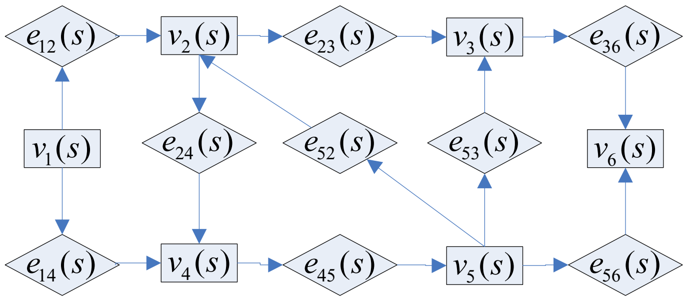

Fig. 2. Frequency-domain spanning graph model of a directed stochastic transportation network.

The capacities of arcs are represented as random variables, and their mathematic graph elements in the frequency domain are shown in Table 1:

Table 1 The parameters of stochastic capacity in the model

Road arc	Capacity	Probability	Road arc	Capacity	Probability
	2	0.25		4	0.1
e_{12}	3	0.5	e_{14}	5	0.7
	4	0.25		6	0.2
e_{24}	1	0.5	e_{52}	1	0.5
	2	0.5		2	0.5
	3	0.2		1	0.2
e_{23}	4	0.6	e_{45}	2	0.5
	5	0.2		3	0.3
	4	0.3		2	0.1
e_{36}	5	0.3	e_{53}	3	0.8
	6	0.4		4	0.1
e_{56}	2	0.5			
	3	0.5			

When the occurrence probabilities of 2, 3, and 4 units' capacity are equal to 0.25, 0.5, and 0.25 respectively, its discrete probability density can be depicted as $e_{12}(s)=0.25e^{-2s}+0.5e^{-3s}+0.25e^{-4s}$.

Similarly, $e_{52}(s)=0.5e^{-s}+0.5e^{-2s}$, $e_{24}(s)=0.5e^{-s}+0.5e^{-2s}$, $e_{53}(s)=0.1e^{-2s}+0.8e^{-3s}+0.1e^{-4s}$, $e_{56}(s)=0.5e^{-2s}+0.5e^{-3s}$, $e_{23}(s)=0.2e^{-3s}+0.6e^{-4s}+0.2e^{-5s}$, $e_{45}(s)=0.2e^{-s}+0.5e^{-2s}+0.3e^{-3s}$, $e_{36}(s)=0\ 3e^{-4s}+0.3e^{-5s}+0.4e^{-6s}$, $e_{14}(s)=0.1e^{-4s}+0.7e^{-5s}+0.2e^{-6s}$.

1) Obtain the incidence matrix of capacity in the frequency domain based on the Fig. 2.

$$
\begin{array}{c c c}
\text{Nodes} & \begin{array}{cc}\text{States of outflow} & \text{States of inflow}\end{array} & \text{Stochastic degree} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{bmatrix}
e_{12}(s)\cdot e_{14}(s) & 1 \\
e_{23}(s)\cdot e_{24}(s) & e_{12}(-s)\cdot e_{52}(-s) \\
e_{36}(s) & e_{23}(-s)\cdot e_{53}(-s) \\
e_{45}(s) & e_{14}(-s)\cdot e_{24}(-s) \\
e_{36}(s)\cdot e_{53}(s)\cdot e_{52}(s) & e_{45}(-s) \\
1 & e_{36}(-s)\cdot e_{56}(-s)
\end{bmatrix} &
\begin{array}{c} E(\Delta_1) \\ E(\Delta_2) \\ E(\Delta_3) \\ E(\Delta_4) \\ E(\Delta_5) \\ E(\Delta_6) \end{array}
\end{array}
$$

Then compute the values of stochastic degree as follows:

$E(\Delta_1)=-e_{12}'(s)\big|_{s=0}-e_{14}'(s)\big|_{s=0}-0=8.1$, $E(\Delta_2)=-e_{23}'(s)\big|_{s=0}-e_{24}'(s)\big|_{s=0}-e_{12}'(-s)\big|_{s=0}-e_{52}'(-s)\big|_{s=0}=1$,

$E(\Delta_3) = -e_{36}^{'}(s)\big|_{s=0} - e_{23}^{'}(-s)\big|_{s=0} - e_{53}^{'}(-s)\big|_{s=0} = -1.9$, $E(\Delta_4) = -e_{45}^{'}(s)\big|_{s=0} - e_{14}^{'}(-s)\big|_{s=0} - e_{24}^{'}(-s)\big|_{s=0} = -4.5$,

$E(\Delta_5) = -e_{56}^{'}(s)\big|_{s=0} - e_{53}^{'}(s)\big|_{s=0} - e_{52}^{'}(s)\big|_{s=0} - e_{45}^{'}(-s)\big|_{s=0} = 4.9$, $E(\Delta_1) = 0 - e_{36}^{'}(-s)\big|_{s=0} - e_{56}^{'}(-s)\big|_{s=0} - 0 = -7.6$.

Finally calculate the capacity of every node v_i with the *conv* command of the MATLAB.

$v_1(s) = e_{12}(s)e_{14}(s) = 0.025e^{-6s} + 0.225e^{-7s} + 0.425e^{-8s} + 0.275e^{-9s} + 0.05e^{-10s}$

Similarly,

$v_2(s) = e_{23}(s)e_{24}(s)e_{12}(-s)e_{52}(-s) = 0.0125e^{2s} + 0.0875e^{s} + 0.2375 + 0.325e^{-s} + 0.2375e^{-2s} + 0.0875e^{-3s} + 0.0125e^{-4s}$,

$v_3(s) = e_{36}(s)e_{23}(-s)e_{53}(-s) = 0.006e^{5s} + 0.072e^{4s} + 0.23e^{3s} + 0.31e^{2s} + 0.28e^{s} + 0.094 + 0.008e^{-s}$,

$v_4(s) = e_{45}(s)e_{14}(-s)e_{24}(-s) = 0.01e^{7s} + 0.105e^{6s} + 0.305e^{5s} + 0.365e^{4s} + 0.185e^{3s} + 0.03e^{2s}$,

$v_5(s) = e_{56}(s)e_{53}(s)e_{52}(s)e_{45}(-s) = 0.005e^{-2s} + 0.0625e^{-3s} + 0.2225e^{-4s} + 0.35e^{-5s} + 0.265e^{-6s} + 0.0875e^{-7s} + 0.0075e^{-8s}$,

$v_6(s) = e_{36}(-s)e_{56}(-s) = 0.15e^{9s} + 0.3e^{8s} + 0.35e^{7s} + 0.2e^{6s}$.

2) Measure the high (low) value of the stochastic positive or negative degrees among nodes using $E(\Delta_i)$. According to the proposed Lemma and the incidence matrix of capacity, we can find the minimum bi-directional cut-set, the minimum positive cut-set and any desired k th minimum cut sets.

2.1) the minimum bi-directional cut-set $(S,\overline{S})_{\min}$ (min-flow) and k th minimum bi-directional cut-set.

It is easy to find the nodes v_3 and v_4 with the stochastic negative degree in node set V (except the origin node v_1 and the destination node v_6), and the value of $E(\Delta_1) + E(\Delta_3) + E(\Delta_4)$ is the minimum, so $(S,\overline{S})_{\min}$ can be obtained as follows:

$S = (v_1, v_3, v_4)$, $\overline{S} = (v_2, v_5, v_6)$

The frequency-domain function of the min-flow

$q(s) = v_1(s)v_3(s)v_4(s) = e_{12}(s)e_{14}(s)e_{36}(s)e_{23}(-s)e_{53}(-s)e_{45}(s)e_{14}(-s)e_{24}(-s) = e_{12}(s)e_{36}(s)e_{23}(-s)e_{53}(-s)e_{45}(s)e_{24}(-s)$

which is identical to the product of the corresponding outflow and inflow states in the 1,3 and 4 row of the incidence matrix.

Similarly, according to the Inference, we get 1th-minimum bi-directional cut-set

$S = (v_1, v_4), \overline{S} = (v_2, v_3, v_5, v_6)$

$q(s) = v_1(s)v_4(s) = e_{12}(s)e_{45}(s)e_{24}(-s)$ equals to the product of the corresponding outflow and inflow states in the 1and 4 row of the incidence matrix;

2th-minimum bi-directional cut-set

$S = (v_1, v_3), \overline{S} = (v_2, v_4, v_5, v_6)$

$q(s) = v_1(s)v_3(s) = e_{12}(s)e_{14}(s)e_{36}(s)e_{23}(-s)e_{53}(-s)$, that is the product of the corresponding outflow and inflow states in the 1and 3 row of the incidence matrix.

2.2) the minimum positive cut-set $(S,\overline{S})^{+}_{\min}$ (max-flow) .

$S = (v_1, v_4)$, $\overline{S} = (v_2, v_3, v_4, v_6)$

$q(s) = v_1(s)v_4(s) = e_{12}(s)e_{45}(s)$, which is identical to the product of the corresponding outflow and inflow states in the 1and 4 row of the incidence matrix, then the $e_{24}(-s)$ in the product equation is removed.

3) According to Equation (8) (9) (10), we acquire the frequency-domain functions $q(s)$ and $Q(s)$ of the network's flow capacities. The frequency-domain functions of the min-flow are as follows,

$q(s) = v_1(s)v_3(s)v_4(s) = e_{12}(s)e_{36}(s)e_{23}(-s)e_{53}(-s)e_{45}(s)e_{24}(-s) = 0.0003e^{-7s} + 0.0049e^{-6s} + 0.0296e^{-5s} + 0.0936e^{-4s} + 0.1832e^{-3s} + 0.2408e^{-2s} + 0.2214e^{-s} + 0.1431 + 0.0629e^{s} + 0.0175e^{2s} + 0.0026e^{3s} + 0.0002e^{4s}$,

$Q(s) = 0.0003e^{-7s} + 0.0052e^{-6s} + 0.0348e^{-5s} + 0.1284e^{-4s} + 0.3116e^{-3s} + 0.5524e^{-2s} + 0.7738e^{-s}$.

The above equations are the frequency-domain probability density and the probability distribution functions of the minimum bi-directional cut-set, i.e. the min-flow; similarly, we can have the frequency-domain functions of the minimum positive cut-set, i.e. the max-flow.

The frequency-domain functions of the max-flow are shown as follows,

$$q(s)=v_1(s)v_4(s)=e_{12}(s)e_{45}(s)=0.075e^{-7s}+0.275e^{-6s}+0.375e^{-5s}+0.225e^{-4s}+0.05e^{-3s},$$

$$Q(s)=0.075e^{-7s}+0.35e^{-6s}+0.725e^{-5s}+0.95e^{-4s}+1e^{-3s}.$$

Similarly, the frequency-domain functions of the 1th min-flow are as follows,

$$q(s)=v_1(s)v_4(s)=e_{12}(s)e_{45}(s)e_{24}(-s)=0.0375e^{-6s}+0.175e^{-5s}+0.325e^{-4s}+0.3e^{-3s}+0.1375e^{-2s}+0.025e^{-s},$$

$$Q(s)=0.0375e^{-6s}+0.2125e^{-5s}+0.5375e^{-4s}+0.8375e^{-3s}+0.975e^{-2s}+1e^{-s}.$$

4) Using MATLAB, we give the following analysis as shown in Fig. 3.

We make a comparison among the probability density curves of the minimum bi-directional cut-set (i.e. min-flow), the 1th-minimum bi-directional cut-set (i.e. second min-flow) and the minimum positive cut-set (i.e. max-flow), which are displayed in Fig.3(a). The contrast of their probability distribution curves is shown in Fig.3 (b). The above-mentioned graphs can be used for the following analyses:

(1) Some analyses are interested in the maximum probability given some predetermined flow cost $\overline{Q}$; its mathematical model is $\max \Pr\{Q \le \overline{Q}\}$;

(2) Some analyses are interested in the max-flow or min-flow which given certain confidence level a ; its mathematical model is $\Pr\{Q \le \overline{Q}\} \ge a$.

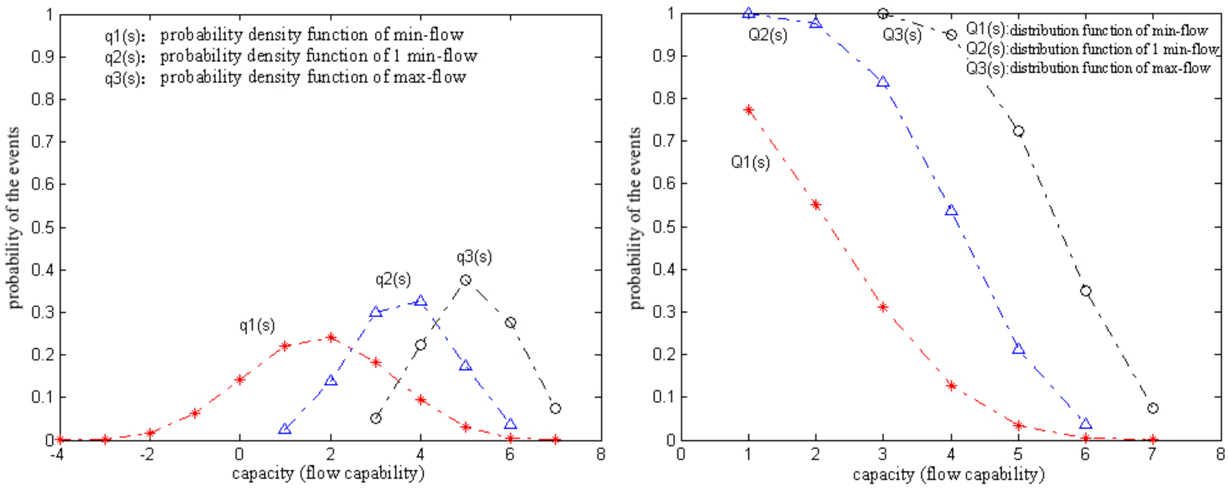

Fig. 3. (a) Probability density graph of flow capacity (b) Probability distribution graph of flow capacity

So when we predetermine the max-flow value which is 6, 5, 4 or 3 respectively and the min-flow value which is 4, 3, 2 or 1, the maximum probability of events is shown in Table 2; when we select different confidence levels α such as 0.7 and 0.3 respectively, the analytical result is shown in Table 3.

Table 2 Analytical result of predetermined max-flow or min-flow value

Maximum Flow Capacity $\overline{Q}$	Probability Pr	Minimum Flow Capacity $\overline{Q}$	Probability Pr
6	0.35	4	0.1284
5	0.725	3	0.3116
4	0.95	2	0.5524
3	1	1	0.7738

Table 3 Analytical result of predetermined confidence level

Confidence Level α	Maximum Flow Capacity $\overline{Q}$	Minimum Flow Capacity $\overline{Q}$
0.9	4	0
0.7	5	1
0.5	5	2
0.3	6	3

The above-mentioned example analysis only illustrates a transportation network with the discrete frequency-domain function. In practice, plenty of convolution calculations will appear as the result of continuous probability densities in the time domain; while the calculation of frequency-domain function only needs multiplication. Moreover, the probability density of the time-domain function is complex; in contrast, its corresponding function in the frequency domain is very simple. For example, the frequency-domain function of the normal distribution is $e^{\frac{1}{2}\sigma_{ij}^2 s^2 - \mu_{ij} s}$, the frequency-domain function of the negative exponential distribution is $\frac{\lambda}{\lambda + s}$, the frequency-domain function of the Γ distribution is $(\frac{\lambda}{\lambda + s})^n$. We can transform a distribution into its frequency-domain function, with the help of Property 1. Therefore, we can conclude that, our model will be more effective for the quantitative analysis of the flow capacity with the continuous frequency-domain function.

5. Conclusion

In this paper, we presented a new approach to the stochastic flow problem. Our contribution could be summarized in the following aspects. First, the stochastic transportation network capacity model and the frequency-domain spanning graph were presented for the first time. In this model, a complex probability distribution in the time-domain could be translated into a simple frequency-domain function. Second, four definitions and one lemma were proposed to help find the stochastic minimum flow and stochastic maximum flow. Third, in order to resolve the stochastic flow problem, a corresponding frequency-domain analysis algorithm was suggested. We use a numerical example to evaluate the feasibility and effectiveness of the above model and algorithm. The results of the experiment show the analysis method is simple and convenient to solve various probability distributions. We hope this work can bring some new ideas and novel tools to analyze stochastic flow problems.

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